A NOTE ON NATURAL TRANSFORMATION OF MORE GENERALIZED HYPERGEOMETRIC FUNCTION

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ABSTRACT

The objective of this paper is to investigate natural transform of the more generalized Hypergeometric function (R –function & S- Function) introduced by Hartley and Lorenzo [4] in which Riemann–Liouville integrals are replaced by more general Prabhakar integrals. We analyze and discuss its properties in terms of Mittag-Leffler functions. Further, we show some applications of these natural transform in classical equations of mathematical physics, like the heat and the free electron laser equations, and in difference-differential equations governing the dynamics of generalized renewal stochastic processes.

Key Words: R-function, S-Function, Natural transform, Riemann Liouville fractional Integrals, Mittag-Leffler Function.

INTRODUCTION

Partial differential equations and their applications arise frequently in many branches of physics, Engineering, and other sciences. There are many works that provided using integral transform method to solve some types of partial differential equations, for example in [06, 7] integral transform is used to solve boundary value problems and integral equations.

Laplace transform is one of the most used in the mathematical and engineering community. Definition of the Laplace transform, notations, and Laplace transforms of some elementary functions can be found in [8].

Furthermore, one dimensional Laplace transform was extended to two dimensional and called as double Laplace transform. The first introduction of double Laplace transform was in [7]. Some operation calculus of double Laplace transform can be found in [9]. Double Laplace transform was used to solve heat, wave, and Laplace's equations with convolution terms (see [10]), telegraph and partial integro differential equations. Sumudu transform was first introduced by [14] and some of its applications were given by [15]. The aims of this study are to generalize the definition of single Natural transform to double Natural transform and achieve its main properties, in order to solve telegraph, wave and partial integro-differential equations.

DEFINITIONS AND PRELIMINARIES USED IN THIS PAPER:

1. ***R*-function:** To affect the direct solution of the fundamental linear fractional order differential equation, the more generalized Hypergeometric function (R –function) introduced by Hartley and Lorenzo [4] defined as follows

$$R_{\mu,v}[a,c,t]=\sum_{n=0}^{\infty}\frac{(a)^n(t-c)^{(n+1)\mu-1-v}}{\Gamma(n+1)\mu-v}, \text{ where } \mu>0 .$$

A more compact notation is given as

$$R_{\mu,v}[a,t-c]=\sum_{n=0}^{\infty}\frac{(a)^n(t-c)^{(n+1)\mu-1-v}}{\Gamma(n+1)\mu-v},$$

which is particularly useful when c=0

S-Function:

The S -function defined and studied by Saxena and Daiya [10] as follows:

$$S_{p,q}^{(\alpha,\beta,\gamma,\tau)}\left[\begin{matrix} a_1,a_2,..a_p \\ b_1,b_2,....b_q \end{matrix}, x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n,(a_2)_n,..........(a_p)_n \quad (\gamma)_{n\tau,k}}{(b_1)_n,(b_2)_n,..........(b_q)_n \quad \Gamma(n\alpha+\beta)} \frac{x^n}{n!}$$

Where, $k \in R, \alpha,\beta,\gamma,\tau \in C$. $R(\alpha) > 0$, $a_1,a_2,..a_p$, $b_1,b_2,....b_q, R(\alpha) > kR(\tau)$ and $p < q+1$. The Pochhammer symbol $(\tau)_\mu$ defined interms of gamma function as follows:

$$(\tau)_\mu = \frac{\Gamma(\tau+\mu)}{\Gamma(\tau)} = \left\{\begin{matrix} 1, & \mu = 0,\tau,\mu \in c \\ \tau(\tau+1)(\tau+2)\ldots\ldots(\tau+\mu-1) \end{matrix}\right\}$$

NATURAL TRANSFORM

In mathematics, the **Natural transform** is an integral transform similar to the Laplace transform and Sumudu transform, introduced by Zafar Hayat Khan[1] in 2008. It converges to both Laplace and Sumudu transform just by changing variables. Given the convergence to the Laplace and Sumudu transforms, the N-transform inherits all the applied aspects of the both transforms. Most recently, F. B. M. Belgacem[2] has renamed it the **natural transform** and has proposed a detail theory and applications.[3][4] The natural transform of a function *f*(*t*), defined for all real numbers $t \geq 0$, is the function *R*(*u*, *s*), defined by:

$$R(u,s) = N[f(t)] = \int_0^\infty e^{-st} f(ut)dt \text{ , } Re(S) > 0, u(-\tau_1,\tau_2)$$

Provided the function $f(t) \in R^2$ *is defined in the set*

$$A = \{ f(t) \,|\, \exists\, M, \tau_1,\tau_2 > 0. |f(t)| < M \; e^{\frac{|t|}{\tau_j}} \}.$$

Khan[1] showed that the above integral converges to Laplace transform when $u = 1$, and into Sumudu transform for $s = 1$.

Lemma-I:

For instance the Natural transform of the t^n, $n > -1$ is given by [3]

$$N[t^n] = \int_0^\infty e^{-st} (ut)^n dt \text{ , } Re(S) > 0,$$

$$= u^n \int_0^\infty e^{-st} t^n dt$$

$$= u^n \frac{\Gamma(n+1)}{s^{n+1}}$$

MAIN RESULTS

In this section, we consider the natural transform of the more generalized hypergeometric function called R-function and making use of the given above lemma to derive following useful results.

Theorem (1.1): Let $A = \{ f(t) \,|\, \exists\, M, \tau_1,\tau_2 > 0. |f(t)| < M \; e^{\frac{|t|}{\tau_j}} \}$, and $N[f(t)]$ be the natural transform associated with R-function . Then there holds the following relationship

$$N\{R_{\mu,v}[a,c,t] = \} = \frac{1}{s} \quad R_{\mu,v}[a,c,t]\left[\begin{matrix} a_1,a_2,..a_p,1 \\ b_1,b_2,....b_q \end{matrix}, x\right]$$

Provided the function $f(t) \in R^2$.

Proof: By using the definition of the generalized S -function of fractional calculus and the natural transform, we get

$$N\{R_{\mu,v}[a,c,t]\} = N\left\{\sum_{n=0}^{\infty}\frac{(a)^n(t-c)^{(n+1)\mu-1-v}}{\Gamma(n+1)\mu-v}\right\}$$

$$N\{R_{\mu,v}[a,c,t]\} = \sum_{n=0}^{\infty}\frac{(a)^n}{\Gamma(n+1)\mu-v}N\left\{(t-c)^{(n+1)\mu-1-v}\right\}$$

By making use of lemma –I in above equation, we get

$$N\{R_{\mu,v}[a,c,t]\} =$$

$$\sum_{n=0}^{\infty}\frac{(a)^n}{\Gamma(n+1)\mu-v}\,u^n\frac{\partial(n+1)}{s^{n+1}}$$

Or

$$N\{R_{\mu,v}[a,c,t]\} = \frac{1}{s}\quad R_{\mu,v}[a,c,t]\begin{bmatrix} a_1,a_2,..a_p,1 \\ b_1,b_2,....b_q \quad , \quad x\end{bmatrix}$$

Theorem (1.2): Let $A=\{\ f(t)\ |\exists\ M,\tau_1,\tau_2>0.|f(t)|<M\ e^{\frac{|t|}{\tau_j}}\}$, and $N[f(t)]$ be the natural transform associated with S-function . Then there holds the following relationship

$$N\left\{\begin{matrix}(\alpha,\beta,\gamma,\tau)\\ \mathbf{S}\\ \mathbf{p,q}\end{matrix}\begin{bmatrix} a_1,a_2,..a_p \\ b_1,b_2,....b_q \quad , \quad x\end{bmatrix}\right\} = \frac{1}{s}\begin{matrix}(\alpha,\beta,\gamma,\tau)\\ \mathbf{S}\\ \mathbf{p+1,q}\end{matrix}\begin{bmatrix} a_1,a_2,..a_p,1 \\ b_1,b_2,....b_q \quad , \quad x\end{bmatrix}$$

Provided the function $f(t) \in R^2$.

Proof: By using the definition of the generalized S -function of fractional calculus and the natural transform, we get

$$N\left\{\begin{matrix}(\alpha,\beta,\gamma,\tau)\\ \mathbf{S}\\ \mathbf{p,q}\end{matrix}\begin{bmatrix} a_1,a_2,..a_p \\ b_1,b_2,....b_q \quad , \quad x\end{bmatrix}\right\} = N\left\{\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,..........(a_p)_n\quad(\gamma)_{n\tau,k}}{(b_1)_n,(b_2)_n,..........(b_q)_n\quad\Gamma(n\alpha+\beta)}\frac{x^n}{n!}\right\}$$

$$N\left\{\begin{matrix}(\alpha,\beta,\gamma,\tau)\\ \mathbf{S}\\ \mathbf{p,q}\end{matrix}\begin{bmatrix} a_1,a_2,..a_p \\ b_1,b_2,....b_q \quad , \quad x\end{bmatrix}\right\} = \sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,..........(a_p)_n\quad(\gamma)_{n\tau,k}}{(b_1)_n,(b_2)_n,..........(b_q)_n\quad\Gamma(n\alpha+\beta)\quad n!}N\{x^n\}$$

By making use of lemma –I in above equation, we get

$$N\left\{\begin{matrix}(\alpha,\beta,\gamma,\tau)\\ \mathbf{S}\\ \mathbf{p,q}\end{matrix}\begin{bmatrix} a_1,a_2,..a_p \\ b_1,b_2,....b_q \quad , \quad x\end{bmatrix}\right\} =$$

$$\sum_{n=0}^{\infty} \frac{(a_1)_n, (a_2)_n, \ldots \ldots \ldots . . (a_p)_n \quad (\gamma)_{n\tau,k}}{(b_1)_n, (b_2)_n, \ldots \ldots \ldots . . (b_q)_n \quad \Gamma(n\alpha+\beta) \quad n!} u^n \frac{\varepsilon(n+1)}{s^{n+1}}$$

Or

$$N\left\{ \begin{matrix} (\alpha,\beta,\gamma,\tau) \\ \mathbf{S} \\ \mathbf{p},\mathbf{q} \end{matrix} \begin{bmatrix} a_1, a_2, ..a_p & \\ b_1, b_2, \ldots . b_q & , \; x \end{bmatrix} \right\} = \frac{1}{s} \begin{matrix} (\alpha,\beta,\gamma,\tau) \\ \mathbf{S} \\ \mathbf{p}+\mathbf{1},\mathbf{q} \end{matrix} \begin{bmatrix} a_1, a_2, ..a_p, 1 & \\ b_1, b_2, \ldots . b_q & , \; x \end{bmatrix}$$

CONCLUSION

This work deals with definition of Natural transform and S-function. Fundamental properties of Natural transform are obtained. Further, some examples and applications on Natural transform are presented. Using Natural transform to solve some types of equations with variable coefficients will be a future work.

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